Algebraic approach to the Hulthen potential

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Abstract

In this paper the energy eigenvalues and the corresponding eigenfunctions are calculated for Hulthen potential. Then we obtain the ladder operators and show that these operators satisfy $SU(2)$ commutation relation.

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1 Introduction

In the recent years, Lie algebraic methods have been the subject of the interest in many of fields of physics. For example the algebraic methods provide a way to obtain wave functions of polyatomic molecules [1, 2, 3, 4, 5, 6]. These methods provide a description to Dunham-type expansions and to force-field variational methods [7]. It is clear that systems displaying a dynamical symmetry can be treated by algebraic methods [8, 9, 10, 11]. To see the ladder operators of a quantum system with some important potentials such as Morse potential the Pöschel-Teller one, the pseudo harmonic one, the infinitely square-well one and other quantum systems refer to [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22].

We know that the symmetry and degeneracy of the states of a system are associated with each other. For example, a system that possesses rotational symmetry is usually degenerate with respect to the direction of the angular momentum, i.e. with respect to the eigenvalues of a particular component. Beyond the degeneracies arising, say, in rotational symmetry there is the possibility of degeneracies of different origin. Such degeneracies are to be expected whenever the Shrodinger equation can be solved in more than one way, either in different coordinate system, or in a single coordinate system which can be oriented in different directions. From our present considerations we should expect these degeneracies to be associated with some symmetry, too. The nature of these symmetries is not geometrical. They are called dynamical symmetries, since they are the consequence of particular forms of the Shrodinger equation or of the classical force law.

In this paper we study the dynamical symmetry for the Hulthen potential, by another algebraic approach. The Hulthen potential [23, 24] is one of the important short-range potentials in physics. This potential is a special case of the Eckart potential [25] which has been widely used in several branches of physics and its bound-state and scattering properties have been investigated by a variety of techniques (see e.g., [26] and references therein). We establish the creation and annihilation operators directly from the eigenfunctions for this system, and that the ladders operators construct the dynamical algebra $SU(2)$.

2 Schrödinger equation with Hulthen Potential

The Hulthen potential has the following form [23, 24, 27]

$$V(r) = -V_0 \frac{e^{-\frac{r}{a}}}{1-e^{-\frac{r}{a}}} \tag{1}$$

where $V_0 = Ze^2$ and a are constant parameters. If the potential is used for atoms, the Z is identified with the atomic number. In order to calculate the energy eigenvalues and the corresponding eigenvalues and the corresponding eigenfunctions, the potential function given by Eq.(1) is substituted into the Schrödinger equation:

$$(-\frac{\hbar^2}{2M}\frac{d^2}{dr^2} - V_0 \frac{e^{-\frac{r}{a}}}{1-e^{-\frac{r}{a}}})\psi_n(r) = E_n\psi_n(r) \tag{2}$$

By change of coordinate as $x = \frac{r}{a}$, and introducing the following parameters

$$\varepsilon_m = \frac{E_m}{V_0}, \qquad \frac{\hbar^2}{2M}a^2 = V_0 \tag{3}$$

Now, we can rewrite Eq.(2) as

$$(\frac{d^2}{dx^2}+\frac{e^{-x}}{1-e^{-x}})\psi_n(x)=-\varepsilon_n\psi_n(x) \tag{4}$$

We would like to consider the bound states with

$$\varepsilon_n=-s^2 \tag{5}$$

We rewrite Eq.(4) by using a new variable of the form $y=e^{-x}$,

$$\{y^2\frac{d^2}{dy^2}+y\frac{d}{dy}+(-s^2+\frac{y}{1-y})\}\psi_n(y)=0 \tag{6}$$

The boundary conditions areas following

$$\phi_n(y)|_{y=0}=0 \tag{7}$$

$$\phi_n(y)|_{y=1}=0 \tag{8}$$

The solution of Eq.(6) are as follow:

$$\begin{aligned}\phi_n(y) &= y^s(1-y)\{A\,{}_2F_1(s+1+\sqrt{s^2+1},s+1-\sqrt{s^2+1},2s+1;y)\\ &+ By^{-2s}\,{}_2F_1(-s+1+\sqrt{s^2+1},-s+1-\sqrt{s^2+1},-2s+1;y)\}\end{aligned} \tag{9}$$

where A, and B are constant coefficients. Using the first boundary condition (7), $B=0$. Now to consider the second boundary condition (8), we expand the hypergeometric function ${}_2F_1(a,b,c;y)$ near the $y=1$ [28]

$$\begin{aligned}&{}_2F_1(s+1+\sqrt{s^2+1},s+1-\sqrt{s^2+1},2s'+1;y)=\frac{\Gamma(2s'+1)\Gamma(\epsilon-1)}{\Gamma(s-\sqrt{s^2+1}+\epsilon)\Gamma(s+\sqrt{s^2+1}+\epsilon)}\\ &{}_2F_1(s+1+\sqrt{s^2+1},s+1-\sqrt{s^2+1},2s'+1;1-y)\\ &+(1-y)^{\epsilon-1}\frac{\Gamma(2s'+1)\Gamma(\epsilon-1)}{\Gamma(s+1+\sqrt{s^2+1})\Gamma(s+1-\sqrt{s^2+1})}\\ &\times{}_2F_1(s-\sqrt{s^2+1}+\epsilon,s+\sqrt{s^2+1}+\epsilon,\epsilon;1-y)\end{aligned} \tag{10}$$

where $\epsilon=2(s'-s)$. Also we have

$${}_2F_1(a,b,\epsilon;1-y)=1+\frac{ab}{\epsilon}(1-y)+\cdots \tag{11}$$

Now we consider the limit $\epsilon\to 0$ of Eq. (10), the first term of (10) in this limit will be finite, if

$$s-\sqrt{s^2+1}=-n,\qquad n=0,1,2,\cdots \tag{12}$$

then we have following relation for the first term of (10) in the limit $\epsilon\to 0$

$$\lim_{\epsilon\to 0}\frac{\Gamma(\epsilon-1)}{\Gamma(\epsilon-n)}=(-1)^{n+1}n! \tag{13}$$

Due to the relation (12), in second term of Eq. (10), $\Gamma(s+1-\sqrt{s^2+1})=(-n)!$ whit is infinite unless $s-\sqrt{s^2+1}=0$, in this case using Eq. (11), we can write the second term of (10):

$$\lim_{\epsilon\to 0,\,y\to 1}(1-y)^{-1}\frac{\Gamma(2s+1)}{\Gamma(s+1+\sqrt{s^2+1})}\{1+\frac{\epsilon(s+1+\sqrt{s^2+1}+\epsilon)}{\epsilon}(1-y)+\cdots\}\propto(1-y)^{-1} \tag{14}$$

Then above relation is divergent when $y \to 1$, then the n could not take the zero value, in another term

$$s - \sqrt{s^2+1} = -n, \qquad n = 1, 2, \cdots \tag{15}$$

Finally we obtain following expression for the wave function

$$\psi_n(y) = N_n y^s (1-y) {}_2F_1(2s+1+n, 1-n, 2s+1; y) \tag{16}$$

where the normalization factor is given by

$$N_n = \{\int_0^1 dy\, y^{2s}(1-y)^2 {}_2F_1^2(2s+1+n, 1-n, 2s+1; y)\}^{-\frac{1}{2}} \tag{17}$$

some values of N_n for different n are given in the table (1). Using Eqs. (5,15) we have

$$\varepsilon_n = -s^2 = -(\frac{n^2-1}{2n})^2 \tag{18}$$

then using Eq. (3), one can determine the energy eigenvalues E_n as

$$E_n = -V_0(\frac{n^2-1}{2n})^2, \qquad n = 1, 2, \cdots \tag{19}$$

in this case the energy level is not equidistant.

3 Ladder operators for the Hulthen potential

In this section we address the problem of finding creation and annihilation operators for the Hulthen wave function (16), namely, we intend to find different operators $\hat{L}_\pm$ with following property:

$$\hat{L}_\pm\, \psi_n(y) = l_\pm\, \psi_n(y) \tag{20}$$

we start by establishing the action of the differential operator $\frac{d}{dy}$ on the Hulthen wave functions

$$\begin{aligned} \frac{d}{dy}\psi_n(y) &= \frac{d}{dy}(N_n y^s (1-y)\, {}_2F_1(2s+1+n, 1-n, 2s+1; y)) \\ &= (\frac{s}{y} - \frac{1}{1-y})\psi_n(y) + N_n y^s (1-y) \frac{d}{dy}({}_2F_1(2s+1+n, 1-n, 2s+1; y)) \end{aligned} \tag{21}$$

To obtain the wanted result, we use the following relations [28]

$$\frac{d}{dx}\, {}_2F_1(a, b, c; x) = \frac{ab}{c}\, {}_2F_1(a+1, b+1, c+1; x) \tag{22}$$

$$\frac{a}{c}\, x\, {}_2F_1(a+1, b+1, c+1; x) = {}_2F_1(a, b+1, c; x) - {}_2F_1(a, b, c; x) \tag{23}$$

$$(a-b)\, {}_2F_1(a, b, c; x) + a\, {}_2F_1(a+1, b, c; x) + b\, {}_2F_1(a, b+1, c; x) = 0 \tag{24}$$

$$(a-b)(1-x)\, {}_2F_1(a, b, c; x) + (c-a)\, {}_2F_1(a-1, b, c; x) - (c-a)\, {}_2F_1(a, b-1, c; x) = 0 \tag{25}$$

Using Eq. (22), we obtain following relation for last term of (21):

$$\frac{d}{dy}({}_2F_1(2s+1+n, 1-n, 2s+1; y)) = \frac{(2s+1+n)(1-n)}{(2s+1)} {}_2F_1(2s+1+(n+1), 1-n+1, 2s+1+1; y) \tag{26}$$

Now we use Eqs.(23,24,25) to satisfy the above relation

$$\frac{d}{dy}({}_2F_1(2s+1+n,1-n,2s+1;y)) =$$
$$\frac{(2s+n+1)}{y}\{\frac{(2s+n)}{(1-y)(2s+2n+1)}\,{}_2F_1(2s+1+(n+1),1-(n+1),2s+1;y)$$
$$+(\frac{(n+1)}{(1-y)(2s+2n+1)}-1)\,{}_2F_1(2s+1+(n),1-(n),2s+1;y)\} \tag{27}$$

By substituting the above realtion in Eq. (21) we obtain

$$\begin{aligned}\frac{d}{dy}\psi_n(y) &= (\frac{s}{y}-\frac{1}{1-y}+\frac{2s+n+1}{y}(\frac{(n+1)}{(1-y)(2s+2n+1)}-1))\psi_n(y)\\ &+ (\frac{(2s+n+1)(2s+n)}{y(1-y)(2s+2n+1)})\frac{N_n}{N_{n+1}}\psi_{n+1}(y)\end{aligned} \tag{28}$$

we can rewrite the above equation in the standard form (20)

$$\begin{aligned}[y(1-y)\frac{d}{dy}+y-s(1-y) &+ (1-y)(2s+n+1)(\frac{(n+1)}{(1-y)(2s+2n+1)}-1)]\\ \times(\frac{2s+2n+1}{2s+n+1})\psi_n(y) &= (2s+n)\frac{N_n}{N_{n+1}}\psi_{n+1}(y)\end{aligned} \tag{29}$$

therefor we have following relation for the creation operator

$$\hat{L}_+ = [y(1-y)\frac{d}{dy}+y-s(1-y)+(1-y)(2s+n+1)(\frac{(n+1)}{(1-y)(2s+2n+1)}-1)]\times(\frac{2s+2n+1}{2s+n+1}) \tag{30}$$

satisfying the equation

$$\hat{L}_+\,\psi_n(y) = l_+\,\psi_{n+1}(y) \tag{31}$$

with

$$l_+ = (2s+n)\frac{N_n}{N_{n+1}} \tag{32}$$

Similarly one can obtain the annihilation operator as

$$\hat{L}_- = [-y(1-y)\frac{d}{dy}-y+s(1-y)+(1-y)(n-1)(1-\frac{(2s+n-1)}{(1-y)(2s+2n-1)})]\times(\frac{2s+2n-1}{n-1}) \tag{33}$$

with the following effect over the wave functions:

$$\hat{L}_-\,\psi_n(y) = l_-\,\psi_{n-1}(y) \tag{34}$$

where

$$l_- = (n)\frac{N_n}{N_{n-1}} \tag{35}$$

We now study the algebra associated to the operators $\hat{L}_+$ and $\hat{L}_-$. Based on results (31,34) we can calculate the commutator $[\hat{L}_+,\hat{L}_-]$:

$$[\hat{L}_+,\hat{L}_-]\psi_n(y) = 2(n+s)\psi_n(y) \tag{36}$$

we can define the operator

$$\hat{L}_0 = (\hat{n}+s) \tag{37}$$

where $\hat{n}$ is the number operator

$$\hat{n}\,\psi_n(y) = n\,\psi_n(y) \tag{38}$$

thus the operator $\hat{L}_0$ has the following eigenvalue

$$l_0 = n + s \tag{39}$$

Thus the operators $\hat{L}_\pm$, $\hat{L}_0$ satisfy the commutation relations:

$$[\hat{L}_-, \hat{L}_+] = 2\hat{L}_0, \qquad [\hat{L}_0, \hat{L}_-] = -\hat{L}_-, \qquad [\hat{L}_0, \hat{L}_+] = \hat{L}_+ \tag{40}$$

Which correspond to the $SU(2)$ group for the Hulthen potential.

4 conclusion

In this paper, we have calculated the exact bound-state energy eigenvalues and the corresponding eigenfunctions of the Hulthen potential. We have shown that the energy level is not equidistant in this case. Then we have obtained the raising and lowering operators and we have shown that $SU(2)$ is the dynamical group associated with the bounded region of the spectrum.

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Tables

n	N_n
1	$\sqrt{3+11s+12s^2+4s^3}$
2	$\frac{1+2s}{2}\sqrt{30+47s+24s^2+4s^3}$
3	$\frac{1+3s+2s^2}{3}\sqrt{105+107s+36s^2+4s^3}$
4	$\frac{3+11s+12s^2+4s^3}{12}\sqrt{252+191s+48s^2+4s^3}$

Table 1: This table shown four normalization factor of Eq. (16).